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The quality of life in the dynamics of economic development

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ABSTRACT The neoclassical economic growth model and its extensions in the fields of environmental economics and endogenous growth theory typically represent welfare as a single argument function of consumption when the models are analytically solved. This simplified welfare specification is narrower than those described in the quality-of-life literature and emphasized by proponents of sustainable development. The purpose of this paper is to analytically solve for the properties of a growth model based on a broader quality-of-life measure. The welfare measure includes two arguments, consumption and the stock of nature capital. This formulation enables an analysis of the consequences of the dynamic tension between conventionally defined economic growth and nature capital preservation. We find that a static model without technical progress yields diverse steady states, stability properties, and comparative statics, while a model with exogenous technical progress exhibits unusual comparative dynamics and balanced growth paths. These unusual outcomes have a number of policy-relevant implications for sustainable development.

1 Introduction

Welfare often is represented as a single argument function of consumption in the neoclassical economic growth model, its endogenous growth extensions, and its applications to economies with linkages to natural/environmental resource sectors. This formulation is particularly likely in studies in which the growth model is analytically solved. Yet, the single argument welfare measure is narrower than those discussed in the quality-of-life literature (for example, Diener and Suh, 1997), studied in some national income accounting research (for example, Mäler, 1991), and formulated for the study of the optimal depletion of natural environments (for example, Barrett, 1992; Krautkraemer, 1985; Krutilla, 1967). The consumption-based welfare specification is also one of the targets of criticism

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of the neoclassical paradigm levied by proponents of sustainable development (for example, World Bank, 1997).

The purpose of this paper is to broaden the quality-of-life representation within the conventional economic growth framework in order to assess whether the broader conception has implications for the system's dynamic behavior. If so, the dynamic effects of the quality-of-life measure itself would introduce another policy-relevant distinction between the terms 'growth' and 'development' as these have become distinguished in the sustainable development literature (for example, The Brundtland Commission, 1987; Daly, 1992; World Bank, 1997).[1]

The exposition of the paper is deliberately kept as parsimonious as possible for the purpose of isolating only the effect of broadening the quality-of-life representation itself—holding constant the many other factors that might otherwise affect the system's dynamics. The economic growth model developed for this purpose uses two production inputs—labor and nature capital—with nature capital conceptualized as a generic renewable resource in the manner of Mäler (1991), Brander and Taylor (1998), and Dasgupta and Mäler (2000). In contrast to the standard Ramsey framework, nature capital yields utility in its natural state, as well as through the consumption channel. This formulation leads to a potential trade-off between conventionally defined economic growth and nature capital preservation, exhibiting the dynamic implications of the defined quality-of-life measure at the most fundamental level.

The modeling approach of the paper shares some common features with Krautkraemer (1985), Mäler (1991), Barrett (1992), and Heal (1998) in the sense that these studies, like ours, employ an economic growth framework, model an environmental–economy linkage, and use a utility specification reflecting the value of preserved nature capital. Mäler's work defines a Hamiltonian for a Ramsey model with a greatly expanded set of arguments in order to elicit the national income accounting implications from the first-order conditions. The Krautkraemer and Barrett line of research examines the impact on the optimal depletion path of a non-renewable resource that offers both amenity and consumption values.[2] Heal's focus is on exploring the implications of alternative definitions of sustainable development within the context of the expanded model with nature capital in the utility specification.

In contrast, the focus of our analysis is on systems dynamics in the presence of non-linear dynamic processes. As such, the emphasis and the setting of our study are closer to the emerging literature in ecological economics on this topic (for example, Perrings and Walker, 1997; Ludwig, Walker, and Holling, 1997; Arrow *et al.*, 2000; Mäler, 2000) than to the

[1] The terms 'welfare' and 'quality of life', 'Ramsey model' and 'neoclassical economic growth model', and 'equilibrium' and 'steady state', are used interchangeably, respectively.

[2] This line of research was begun by Krutilla (1967). See also Hartman (1976) for a related extension that involves determining the optimal rotation age for a standing forest yielding amenity value in its uncut condition, as well as direct consumptive value from harvest.

conventional economic growth literature. In fact, one contribution of the paper is to demonstrate how dynamic complexity can arise within the economic component of a growth model itself, offering another potential channel for dynamic complexity in the system—and one that could potentially augment the dynamic diversity generated through complex ecological interactions. The conventional economic growth literature (for example, see Aghion and Howitt, 1998 for a review) and its extensions to natural resources (for example, see Toman, Pezzey, and Krautkraemer, 1995 for a review) have largely abstracted from the possibility of dynamic complexity in the macro-economy.

To preview the paper's results, the growth model with a broadened quality-of-life measure exhibits a number of non-trivial departures from the standard case. A model without technological progress can exhibit multiple equilibria, with different welfare levels. There are three types of steady state in the model: saddle-point stable nodes, unstable nodes, and non-isolated critical points. With multiple steady states, the economy does not necessarily converge to the same unique steady state, as occurs in standard economic growth models. Moreover, the possibility of multiple steady states in an economic growth framework implies the possibility of 'non-resilient economic systems' analogous to those identified for ecosystems in the ecology and the ecological economics literatures (for example, Perrings and Walker, 1997; Ludwig, Walker, and Holling, 1997; Arrow *et al.*, 2000; Mäler, 2000).

The second departure from the standard case is the fact that increases in the rates of population growth and nature capital depreciation exhibit unusually diverse initial consequences on consumption, nature capital, and the quality of life. In the Ramsey model, these parametric increases unambiguously lower consumption and welfare. However, in the model developed in this paper, comparative static results are contingent on the type of steady state and its position on the nature capital domain. The quality of life unambiguously declines initially only in the case where the equilibrium from which the move is made is a saddle-point stable node located at a nature capital level less than the maximum growth point. The welfare comparative statics result is ambiguous if the initial move is from a saddle-point stationary state to the right of the maximum growth point, and *increasing* if the initial move is from an unstable node.

Third, the model with exogenous labor-augmenting technological progress generates unusual dynamic behavior. The economy can display multiple balanced growth paths for consumption, nature capital, and welfare, rather than the single path displayed in models that build on the standard case, such as the seminal natural resource extension of the Ramsey model developed by Stiglitz (1974). This has the important policy implication that equivalent rates of technological progress will not close the wealth gap between two countries with identical parameters but different initial conditions. Exogenous wealth transfers, rather than technology development and trade, would be needed to overcome the path dependency in this case, that is, to induce global convergence to a common quality-of-life level. Another result of the model is that the rate of technological progress can initially lower the quality of life along the transition path, in contrast to the standard case in which welfare smoothly increases (for example, Stiglitz, 1974). Thus, technology has a more varied and possibly less propitious short-run effect in a growth model in which the quality-of-life measure is more broadly defined.

In summary, the system dynamic manifestation of the distinction between 'sustainable development' and 'sustainable growth' turns out to be theoretically significant at many levels. Presumptively, such dynamic diversity has methodology and policy implications. For example, the results of the paper at least raise a question about the accuracy of estimates generated by conventional computable general equilibrium models of the economic effects of global climate change or climate control policies, since these models, in the main, rely on the traditional Ramsey growth framework for their theoretical foundation (for example, see Jorgenson and Wilcoxen, 1991). More generally, the diverse dynamic effects demonstrated in the paper suggest that policy formulation to promote sustainable development may face more complex challenges than is commonly recognized by conventional economic growth modelers.

The paper is organized as follows. Section 2 describes the assumptions and structure of the model for an economy without technological progress. Section 3 analyzes the existence and stability of the equilibria that result from this model, and computes comparative statics. Section 4 analyzes the dynamics when exogenous technological progress is added to the model. Section 5 discusses the implications of the analysis, and suggests future research.

2 The model without technological progress

To isolate for the effect of broadening the quality-of-life measure, the mathematical structure of a conventional Ramsey economic growth model is maintained with only one deviation: nature capital is added as an argument to the utility function. If policy-relevant dynamics arise within this formulation, they are likely to become more consequential in more complex, empirically realistic specifications. Hence, the approach is to initiate the study of the relationship between the quality of life and dynamic complexity at its most fundamental level.

Macro-level economic growth models with natural resources typically employ two functional forms for the resource growth rate of a renewable resource: linear (Mäler, 1991) and logistic (Brander and Taylor, 1998). The adapted Ramsey equation of motion used in this paper is consistent with the second approach. Let S denote the stock of nature capital in time t (time subscripts are suppressed in the entire presentation for notational convenience). Nature capital is construed broadly as a generic renewable resource, in the manner of other macro-level studies (Mäler, 1991; Brander and Taylor, 1998; Dasgupta and Mäler, 2000).[3] Without resource consumption, nature capital evolves according to

[3] The aggregate nature capital formulation found in macro-economic models is likely to lead to biases in view of the heterogeneity of natural capital in the real world. Further study is needed to understand the consequences of such biases.

$$\frac{dS}{dt} = G(S,L) - bS \tag{1}$$

$G(S,L)$ is the instantaneous gross growth rate of nature capital, and b is the rate at which nature capital naturally degrades.[4] The possibility of augmenting resource productivity with labor input follows Mäler (1991), and is consistent with Dasgupta and Mäler (2000). Resource augmentation can be construed quite broadly, for example, as the recovery of a degraded resource, the improved management of an existing resource, or the discovery of a new resource.[5]

$G(S,L)$ is assumed to be twice continuously differentiable, strictly concave and non-decreasing in L and S.[6] To avoid degeneracy, $G(S,L_o) - bS > 0$ for $S \in (0,S^+)$, where S^+ is the solution of $G(S^+, L_o) - bS^+ = 0$, and $L_o \equiv L(0)$ is the initial labor force.[7]

With L taken as a parameter, and assuming the restrictions noted, equation (1) has the same mathematical structure as the equation of motion derived from the logistic growth equation. It has $dS/dt = 0$ at $S = 0$ and $S = S^+$ and, with $G_{SS} < 0$, exhibits a maximum growth rate at S_{msy} on $(0,S^+)$ where $\partial(dS/dt)/\partial S = G_S(S_{msy}) - b = 0$. Since $\partial(dS/dt)/\partial S < 0$ only for $S \in (S_{msy}, S^+]$, the first equilibrium at $S = 0$ is unstable while the second at S^+ is stable. As a result, the steady-state level for S without any resource consumption is likely be at S^+, as in the logistic formulation.[8]

Nature capital, S, yields consumption flow, C, in proportion to the rate at which humans deplete S. Without loss of generality, the proportionality constant linking the depletion of nature capital to consumption is taken to be 1. Hence, equation (1) can be modified to take into account the consumption effect on the growth of nature capital as follows

$$\frac{dS}{dt} = G(S,L) - bS - C \tag{2}$$

Equation (2) reflects the assumption that consumption only imposes an investment opportunity cost, in the form of foregone resource accumulation.

The other state equation in the model is for human population growth.

[4] It is possible to subsume the term '$-bS$' within the G function. We maintain the separable formulation since it is the standard representation in the Ramsey literature. Again, we wish to minimize deviations from the standard formulation.

[5] This general approach follows Mäler (1991) and Dasgupta and Mäler (2000). Note that Dasgupta and Mäler assume that the resource-augmenting input is a generic composite, that is, expenditure.

[6] The sign of G_L follows Mäler (1991) and is consistent with Dasgupta and Mäler (2000).

[7] As in the neoclassical model and its many derivatives, labor has no other use. For a model where labor is also used in resource extraction, see Krutilla and Reuveny (2000b).

[8] Holding L constant, the logistic formulation and equation (1) exhibit equivalent properties through the second derivative level with respect to S. The logistic formulation does not independently identify resource growth and depreciation as in (1), since these rates are subsumed within the term for the resource growth rate.

Following the standard assumption of the economic growth literature, population grows at a constant rate n:

$$\frac{dL}{dt} = nL \tag{3}$$

Population is assumed to be fully employed, so the terms 'population' and 'labor' can be used interchangeably. As suggested by Dasgupta and Mäler (2000), the customary assumption of exogenous population growth rate is restrictive in view of evidence that population growth rate also has an endogenous component. The customary population assumption is maintained in this paper for the purpose of comparability to the standard growth literature. We return to the issue of endogenous population growth rate in section 5.

Turning to the welfare function, nature capital provides value in its natural state, but also yields utility from depletion in the form of consumption. The representative agent's welfare is assumed to be separable in nature capital and consumption: $W(c,s) = U(c) + V(s)$, with $c \equiv C/L$ and $s \equiv S/L$. The assumption $W_{cs} = 0$ is used in many studies (for example, Heal, 1998; Barrett, 1992; and Fisher, Krutilla, and Cicchetti, 1972) for lack of a theoretical prior about the sign of W_{cs}, as well as for analytic convenience. We maintain this assumption here.[9] As usual, we also assume that U and V are twice continuously differentiable, increasing, strictly concave, and $U_c \rightarrow \infty$ as $c \rightarrow 0$.

The agents in the model derive utility from nature capital per capita. This is a relatively restrictive assumption, implying that nature capital in the model is an excludable resource that could be privatized. Our approach is congruent with the standard assumption in the economic growth literature that property rights are fully specified and costlessly enforced—an assumption also maintained here. Of course, some natural resources may take the form of non-rival, non- excludable public goods. The narrower, per capita specification is maintained here, again, for the sake of parsimony and comparability to the conventional economic growth literature.[10]

Denoting ρ as the positive social rate of discount, the task of the representative agent, under the previously noted assumptions and conditions, can be formalized as the solution to the following dynamic optimization

[9] Presumptively, assuming $W_{cs} \neq 0$ would complicate the dynamic behavior, since fewer terms will cancel, resulting in a more complex system of differential equations.

[10] The public goods formulation would yield a non-autonomous optimal control problem in a model such as ours with growing population. Adding externalities, or other realistic features of the natural capital aggregate, would presumptively complicate the system's dynamics. We discuss this issue in section 5.

$$\max \int_0^{\infty} [U(C/L) + V(S/L)]e^{-\rho t}dt$$
$$s.t.$$
$$\frac{dS}{dt} = G(S,L) - bS - C \tag{4}$$
$$\frac{dL}{dt} = nL$$

$$S(0) > 0, L(0) > 0; S(t) > 0\ C(t) \geq 0, ® \ t$$

The control variable is $c \equiv C/L$, and the state variables are S and L.[11]

The optimal control problem in (4) can be simplified to one state variable, s, and one control variable, c, under the assumption that $G(S,L)$ is homogenous of degree 1. The first step is to use the definition $s \equiv S/L$ to express the left side of Equation (2) as

$$\frac{dS}{dt} = \frac{d(Ls)}{dt} = \frac{dL}{dt}s + \frac{ds}{dt}L \tag{5}$$

Substituting (5) into (2), and using (3), a per capita resource function can be expressed as

$$\frac{ds}{dt} = g(s) - (b + n)s - c \tag{6}$$

where $g(s) \equiv G(S/L,1)$. Following this translation, (4) is rewritten in per capita terms as

$$\max \int_0^{\infty} [U(c) + V(s)]e^{-\rho t}dt$$
$$s.t.$$
$$\frac{ds}{dt} = g(s) - (b + n)s - c \tag{7}$$

$$s(0) > 0; s(t) > 0, c(t) \geq 0, ® \ t$$

The current value Hamiltonian for this problem is

$$H = U(c) + V(s) + \lambda(g(s) - (n + b)s - c) \tag{8}$$

where λ is the shadow price of s. The first-order conditions are

$$H_c = 0 \Rightarrow U_c(c) = \lambda \tag{9}$$

$$\frac{d\lambda}{dt} = \rho\lambda - H_s \Rightarrow \frac{d\lambda}{dt} = -V_s(s) + \lambda(\rho + b + n - g_s(s)) \tag{10}$$

and the transversality condition is $\lim_{t->\infty} [e{-}\rho^t\lambda(t)s(t)] = 0$.[12] The second-order conditions for a maximum are assumed to hold since the functional

[11] The model in (4) reflects the usual assumption in the economic growth literature that the consumption control variable is not subject to a physical bound.

[12] The literature provides examples in which the transversality condition is not necessary for optimization. Nonetheless, in infinite horizon problems with

forms in the model can satisfy them. Equation (9) is the Ramsey optimality condition, equating the marginal utility from consumption per capita to the shadow price of the stock of nature capital per capita. Equation (10) implies that the social rate of discount, ρ, equals the total rate of return on the stock of nature capital per capita, reflecting its net growth, $g_s - (n + b)$, nature capital valuation change, $(d\lambda/dt)/\lambda$, and the ratio of marginal utilities, V_s/U_c.

The next step is to reduce the dimensionality of the model from three variables (c, s, λ) to two variables (c, s). Taking the time derivative of equation (9) and using this result and equation (9) itself to eliminate $d\lambda/dt$ and λ in equation (10) yields

$$U_{cc}(c)\,\frac{dc}{dt} = -\,V_s(s) + U_c(c)\cdot(\rho + b + n - g_s(s)) \tag{11}$$

Equations (6) and (11), the initial conditions, and the transversality conditions define a dynamic system whose solution gives the optimal time path of per capita resource stock and consumption.

The steady states of the system of equations (6) and (11), if they exist, can be investigated by setting ds/dt and dc/dt in (6) and (11) to 0, respectively. This gives the following two equations, hereafter referred to as the *cc* locus and the *ss* locus, respectively:

$$\rho = g_s(s) - (b + n) + \frac{V_s}{U_c} \quad cc\ locus \tag{12}$$

With $U_c(0) = \infty$ and $s \in (0,s^+)$, V_s/U_c drops out of (12) when $c = 0$. Thus, the equation defining the origin of the *cc* locus (or its intercept with the *s* axis) is

$$c = g(s) - (b{+}n)s \quad ss\ locus \tag{13}$$

$$\rho = g_s(s') - (b + n) \tag{14}$$

where s' is the per capita nature capital satisfying equation (14). The slope of the *cc* locus can be obtained by taking differentials of equation (12) and solving for the derivative dc/ds. This gives the expression

$$\frac{dc}{ds} = \frac{U_c(c)\cdot g_{ss}(s) + V_{ss}(s)}{U_{cc}(c)\cdot(\rho + b + n - g_s(s))} \tag{15}$$

The numerator in (15) is negative for all combinations of s and c since g_{ss} and V_{ss} are assumed to be negative while the marginal utility, U_c, is assumed to be positive. In the denominator, U_{cc} is negative. From equation (12), the term $\rho + b + n - g_s$ must be positive since V_s/U_c

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discounting, this condition is often used as a sufficient condition (Barro and Sala-i-Martin, 1995). The transversality condition holds at steady state $\{c^*,s^*,\lambda^*\}$, if it exists, since λ^* and s^* are constant, and $\lim_{t\to\infty} e^{-\rho t} = 0$ for $\rho > 0$. From (9) and the assumption $U_c(0) = \infty$, the condition is not satisfied at $c = 0$. Hence, the transversality condition is consistent with an interior solution for c (that is, $c > 0$). Since c is obtained from the resource, $c > 0$ implies $s > 0$, or resource extinction is not optimal.

> 0 for $c > 0$. Hence the sign of the slope of the *cc* locus in (15) is positive.[13]

Although the first derivative of the *cc* locus can be signed, equation (15) does not contain enough information for making judgements about the second derivative, since the latter will contain the third-order derivatives V_{sss}, U_{ccc}, and g_{sss}, about which standard economic theory offers no guidance. Hence, the possibility of oscillating signs for the second derivative for the *cc* locus cannot be theoretically overruled. This implies that (12) and (13) could intersect at a number of points, opening the door to the possibility of multiple equilibria.

3 System dynamics and comparative statics

Existence and stability of equilibria

The appendix formally proves the existence of at least one equilibrium in the dynamic system represented by equations (6) and (11), and the fact that if multiple equilibria exist, at least one of them will be located on $(0, s_{msy})$, since there can only be one equilibrium on (s_{msy}, s^+). The stability properties of the equilibria can be investigated by linearizing the system around the steady states, assuming that the linearized system is non-singular (Boyce and DiPrima, 1997). If the system is singular, stability properties can be investigated with the phase diagram of the non-linearized system. It is convenient to conduct both kinds of analyses here.

The elements of the Jacobean of the system of equations represented by (6) and (11) are: $J_{11} = \rho + b + n - g_s$, $J_{12} = -U_c\, g_{ss}/U_{cc} - V_{ss}/U_{cc}$, $J_{21} = -1$, and $J_{22} = g_s - (b + n)$, where J_{mn} is an element placed in line m and column n. Solving for the determinant (det(J)) and its trace (tr(J)) yields

$$\det(J) = (g_s - (n + b))\cdot(\rho + b + n - g_s) - \frac{V_{ss}}{U_{cc}} - \frac{U_c g_{ss}}{U_{cc}} \qquad \mathrm{tr}(J) = \rho \quad (16)$$

It is clear that tr(J) > 0 for $\rho > 0$, while the sign of det (J) varies as follows

$$\det(J) < 0 \Rightarrow g_s - (n + b) < \frac{U_c g_{ss} + V_{ss}}{U_{cc}(\rho + b + n - g_s)}$$

$$\det(J) = 0 \Rightarrow g_s - (n + b) = \frac{U_c g_{ss} + V_{ss}}{U_{cc}(\rho + b + n - g_s)} \quad (17)$$

$$\det(J) > 0 \Rightarrow g_s - (n + b) > \frac{U_c g_{ss} + V_{ss}}{U_{cc}(\rho + b + n - g_s)}$$

The left-hand side of (17) is the slope of the *ss* locus, while right-hand side is the slope of the *cc* locus. We denote these respectively as *dc/ds* (*ss*) and *dc/ds* (*cc*). With this notation, (17) implies that det (J) < 0 at equilibria

[13] Note that $g_s(s') = (b + n) + \rho$. Hence, $s' > 0$ and the system cannot have a steady state at ($s = 0$), which, again, means that resource extinction is not optimal in the model.

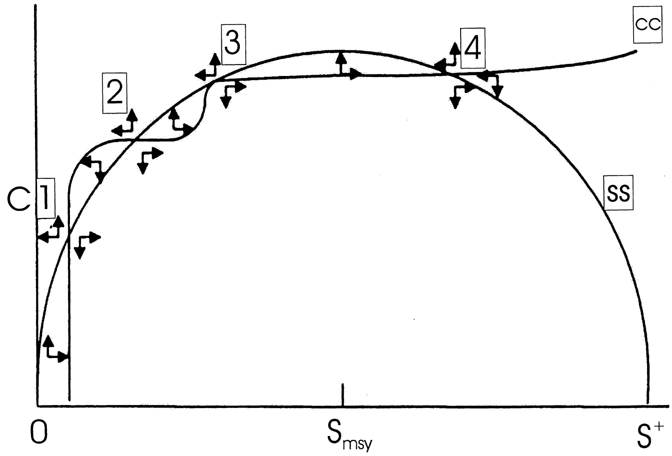

Figure 1.

where dc/ds $(ss) < dc/ds$ (cc), (case I); det $(J) = 0$ where dc/ds $(cc) = dc/ds$ (ss) (case II), and det (J) > 0 where dc/ds $(ss) > dc/ds$ (cc) (case III). In conjunction with tr(J), these relationships define the stability properties of the equilibria. Case I-type steady states are saddle-point stable; case II-type steady states are non-isolated critical points; case III-type steady states are unstable nodes (Boyce and DiPrima, 1997).

A phase diagram can be used to illustrate the properties of the steady states in the system. From (6), $d(ds/dt)/dc = -1 < 0$ implying that per capita nature capital is rising (falling) below (above) the *ss* locus. From (11), $d(dc/dt)/ds = -V_{ss}/U_{cc} - (U_c/U_{cc})\, g_{ss} < 0$, implying that per capita consumption is falling (rising) for levels of nature capital greater (less) than those on the *cc* locus. Figure 1 puts these directional tendencies together in a phase diagram that encompasses the full range of possible equilibria reflected in (17). Equilibria 1 and 4 are saddle-point stable; equilibrium 2 is an unstable node, while equilibria 3 is a non-isolated critical point that has one stable arm in one quadrant, and an unstable arm in each of the three other quadrants. The system in figure 1 will collapse to an alternating sequence of saddle points and unstable nodes (equilibria 1, 2, and 4) if a tangency point does not occur. However, the tangency condition in (17) cannot be mathematically ruled out at the level of general functional forms.

The equilibria implied by (6) and (11) can be contrasted with the Ramsey-based model which has a unique saddle-point stationary state on $(0, s_{msy})$. Economies with the same parameter configuration in the standard case will converge to this stationary state (with the appropriate application of the control variable) notwithstanding differences in the initial conditions. In an economy with nature capital in the utility function, however, there may be a number of stationary states, implying the possibility of path dependency leading to systematic welfare differences in economies with the same parameter configurations.

Moreover, the issue of system's resiliency analyzed in the ecological and ecological-economics literatures has saliency in the context of figure 1. For example, a system with alternating saddle-point stable nodes and unstable nodes will have alternating 'domains of attraction' in the direction of stable arms around the saddle-point stable nodes. A shock to a system at rest at one saddle-point stable equilibrium that bumps the system in the direction of a new stable arm, but out of the current domain of attraction, may move the system to another saddle-point stable node. The new equilibrium could have a lower welfare level than the original. Shocks that move the system in the direction of unstable arms generally have unpredictable consequences. One could well imagine such changes in the economic system interacting with similar changes in the ecosystem, for example, as discussed by Arrow *et al*. (2000) and Mäler (2000) for phosphate concentrations in lakes, and by Perrings and Walker (1997) and Ludwig, Holling, and Walker (1997) for rangeland grazing in savannah systems. The possibility of dynamic complexity in the economic component of the system thus reinforces the concern raised by ecological economists about the welfare implications of dynamic complexity in the ecological component of the system.

Comparative statics

This subsection investigates the comparative statics in the neighborhood of steady states with respect to the social rate of discount (ρ), the degradation rate of nature capital (b), and the rate of population growth (n). Performing the comparative statics computations yields

$$\frac{\partial c}{\partial n} = \frac{\partial c}{\partial b} = \frac{-(U_c/U_{cc})\cdot(g_s - (b+n)) + (s/U_{cc})\cdot(V_{ss} + U_c g_{ss})}{\det(J)} \tag{18}$$

$$\frac{\partial c}{\partial \rho} = \frac{-(U_c/U_{cc})\cdot(g_s - (b+n))}{\det(J)} \tag{19}$$

$$\frac{\partial s}{\partial n} = \frac{\partial s}{\partial b} = \frac{s(\rho + b + n - g_s) - U_c/U_{cc}}{\det(J)} \tag{20}$$

$$\frac{\partial s}{\partial \rho} = \frac{-(U_c/U_{cc})}{\det(J)} \tag{21}$$

The comparative statics designate the initial tendency of the system to move away from steady state, once some parameter is changed, *ceteris paribus*. These effects are inferred from (18)–(21). A summary of the results, and comparison to the standard model without resources in utility, are presented in table 1. Propositions and proofs are provided in the appendix.[14]

[14] Comparative static results are undefined for non-isolated critical points since det(J) is 0, implying that the linearized differential equations system is singular in this case.

Table 1. *Ramsey model versus expanded quality-of-life model, static technology (summary of comparative static effects)**

	Standard Ramsey model	*Model with expanded quality-of-life measure*
Around Saddle Points $(0, \boldsymbol{s}_{msy})$		
Consumption	$\partial c/\partial b < 0$, $\partial c/\partial n < 0$, $\partial c/\partial \rho < 0$	$\partial c/\partial b < 0$, $\partial c/\partial n < 0$, $\partial c/\partial \rho < 0$
Nature capital	$\partial s/\partial b < 0$, $\partial s \partial n < 0$, $\partial s/\partial \rho < 0$	$\partial k/\partial b < 0$, $\partial s/\partial n < 0$, $\partial s/\partial \rho < 0$
Welfare	$\partial W/\partial b < 0$, $\partial W/\partial n < 0$	$\partial W/\partial b < 0$, $\partial W/\partial n < 0$
Around saddle points $(\boldsymbol{s}_{msy,}\ \boldsymbol{s}^{+})$		
Consumption	NA	$\partial c/\partial b < 0$ or > 0, $\partial c/\partial n < 0$ or > 0, $\partial c/\partial \rho > 0$
Capital	NA	$\partial s/\partial b < 0$, $\partial s/\partial n < 0$, $\partial s/\partial \rho < 0$;
Welfare	NA	$\partial W/\partial b < 0$ or > 0, $\partial W/\partial n < 0$ or >0
Around Unstable nodes		
Consumption	NA	$\partial c/\partial b > 0$, $\partial c/\partial n > 0$, $\partial c/\partial \rho > 0$
Capital	NA	$\partial s/\partial b > 0$, $\partial s/\partial n > 0$, $\partial s/\partial \rho > 0$
Welfare	NA	$\partial W/\partial b > 0$, $\partial W/\partial n > 0$
Around non-isolated Critical points		Undefined

Notes: * Results derived in the appendix.
NA = not applicable

The differences in the comparative statics between the two models emerge through two channels. The model with nature capital in utility can exhibit one saddle-point stable equilibrium on the domain (s_{msy}, s^{+}), and multiple equilibria of three types on the domain (0, s_{msy}). In the (s_{msy}, s^{+}) range, the comparative static effects of changes in δ and b on consumption are ambiguous in the model with nature capital in utility, in contrast to the Ramsey model, where consumption initially declines with increases in n or b for the single saddle point occurring on (0, s_{msy}). The comparative statics of the social rate of discount also are different from the standard case. Consumption initially increases with a rise in the discount rate for the saddle point on (s_{msy}, s^{+}), in contrast to the standard model, where consumption declines. This result reflects the fact that depleting nature capital in this range—the initial response to the rise in discount rate—increases consumption possibilities because $g_s - (n + b)$ is negative on (s_{msy}, s^{+}).

Around unstable nodes, the expanded model with nature capital in utility manifests the seemingly perverse result that consumption and nature capital initially rise with population growth rate, capital depreciation, and discount rate. This result can be deduced from figure 1. Comparative static changes in these parameters shift down the *ss* locus, and/or shift the *cc* locus leftward. Either of these directional moves will increase consumption and nature capital around unstable nodes.

The differences in the welfare effects of parametric changes between the model with nature capital in utility and the Ramsey model (again see table 1) reflect both the differences in the comparative static effects on nature capital and consumption just noted, and the fact that the expanded model has a two-argument utility function, with utility monotonically increasing in both consumption and nature capital. Thus

$$\frac{\partial W}{\partial n} = U_c \frac{\partial c}{\partial n} + V_s \frac{\partial s}{\partial n} \tag{22}$$

$$\frac{\partial W}{\partial b} = U_c \frac{\partial c}{\partial b} + V_s \frac{\partial s}{\partial b} \tag{23}$$

The welfare comparative statics follow from (22) and (23) and the results indicated in table 1. The most significant distinction between the two models is the fact that the comparative static effects for welfare in the model with nature capital in utility can be ambiguous on (s_{msy}, s^{+}), while the welfare effects in the vicinity of unstable nodes are unambiguously positive.

In all, these results demonstrate that a relatively small departure from the conventional assumptions in the economic growth literature can dramatically alter the initial system responses to changes in the system's parameters.

4 Technological progress

In this section, technological progress is introduced into the model.

Propositions and proofs are provided in the appendix. The growth rate for nature capital is modified to

$$\frac{dS}{dt} = G(S,AL) - bS - C \tag{24}$$

where $G(\cdot)$, S, L, b and C retain their previous definitions. The variable A denotes labor-augmenting (Harrod-neutral) technology. Technological progress is modeled by the growth of A over time. $G(\cdot)$ is assumed to have the same properties as before. It is also assumed $G(S, A_oL_o) - bS > 0$ for $S \in [0, S^+]$, where $S^+ \equiv G(S^+, A_oL_o) - bS^+ = 0$, with $A_o \equiv A(0)$ being the start-point technology parameter, and L_o defined as before.

Technological progress is assumed to grow at a given rate, α, as follows

$$dA/dt = \alpha A \tag{25}$$

Equation (25) implies that technological progress is exponential. This modeling approach is mostly associated with Solow (1956) and, in the context of exhaustible resources, Stiglitz (1974) and Solow (1974). However, it is also used in some recent studies.[15]

Of course, how to model technologic progress is debated. Endogenous technological progress is supported by such writers as Aghion and Howitt (1998) and Dasgupta and Mäler (2000), the later suggesting that exogenous progress implies the unrealistic assumption 'that the economy is guaranteed a "free lunch" forever (pp. 20)'.[16] Others argue that the explanatory gain from the endogenous progress framework is small (for example, Pack, 1994; Solow, 1994). We believe that the evolution of variables driven by exogenous progress are best regarded as a benchmark for comparison, and not necessarily as a realistic prediction. Nevertheless, we maintain the exogenous progress assumption here for comparability to the standard model, and because our focus is on the effects, rather than the determinants, of progress.

As in section 3, human population is assumed to grow exogenously at the constant rate n. Again given $\rho > 0$, a representative agent is assumed to maximize the sum of discounted utilities from $t = 0$ to $t = \infty$, by choosing $c \equiv C/L$ at each point in time. This optimal control problem can be formalized as

$$\max \int_0^{\infty} [U(C/L) + V(S/L)]\exp(-\rho t)dt$$

$$s.t.$$

$$\frac{dS}{dt} = G(S,AL) - bS - C$$

$$\frac{dL}{dt} = nL \tag{26}$$

$$\frac{dA}{dt} = \alpha A$$

$$S(0) > 0,\ L(0) > 0\ A(0) > 0;\ S(t) > 0,\ C(t) \geq 0,\ \circledR\ t$$

15 Stokey (1998), for example, applies it in a model of economic growth with pollution.

16 A different approach is taken by Boserup (1981) and Simon (1996) who argue that progress is, in effect, caused by population growth.

The problem is simplified to include one state variable under the assumption that $G(\cdot)$ is homogenous of degree 1. To that effect, define E as 'effective labor', where $E \equiv A \cdot L$. Using E, the resource regeneration function becomes $G(S,E)$. Next, define $x \equiv S/E$ as resource stock per unit of effective labor, and $y \equiv C/E$ as consumption per unit of effective labor. As before, $c = C/L$ is per capita consumption and $s = S/L$ is per capita resource stock. Using these definitions, (24) can be written as

$$\frac{dS}{dt} = \frac{d(ALx)}{dt} = \frac{dA}{dt} Lx + \frac{dL}{dt} Ax + \frac{dx}{dt} AL \tag{27}$$

Substituting (27) for dS/dt in (24) and noting that $(dL/dt)/L = n$ and $(dA/dt)/A = \alpha$ gives

$$\frac{dx}{dt} = g(x) - (b + n + a)x - y \tag{28}$$

where $g(x) \equiv G(S/E,1)$ denotes the growth of nature capital as a function of units of effective labor. The mathematical structure of (28) is equivalent to the structure of (6). Thus, our approach is to solve the optimal control problem expressed in units of effective labor, and then to restate its solution in per capita terms.[17] The formal problem can be stated as follows

$$\max \int_0^\infty [U(y) + V(x)]e^{-\rho t} dt$$
$$s.t. \tag{29}$$
$$\frac{dx}{dt} = g(x) - (n + b + \alpha)x - y$$
$$x(0) > 0;\ x(t) > 0,\ y(t) \geq 0 \ \text{®}\ t$$

As in section 3, the transversality condition and the second-order conditions are assumed to hold. The first-order conditions result in two differential equations, one for y and the other for x. The intersection of the xx locus (the locus along which $dx/dt = 0$) and the yy locus (the locus along which $dy/dt = 0$) defines the equilibria in the $x - y$ plane. The solution properties are summarized in table 2. Propositions and proofs are provided in the appendix.

The system represented in (29) exhibits a path in the $c - s$ plane in which s and c grow at the rate of technological progress (α). Although the 'long-run' is the same in the model with nature capital in utility and in the Ramsey model, in the sense that in both cases s and c grow at the rate of

[17] For a similar approach, see Stiglitz (1974).

Table 2. *Ramsey model versus expanded quality-of-life model, technological progress*

	Standard Ramsey model	*Expanded quality-of-life model*
Balanced growth path	Single	Single Multiple
Transition path	Smooth rise in c, s, and W	Initial decrease in s Initial increase or decrease in c Initial increase or decrease in W
Long-run welfare growth rate	Rate of technological progress	Rate of technological progress

technological progress, the transitional dynamics are different. To see that, note that with α entering (28) in the same way as b and n, and with (28) in $x - y$ space otherwise equivalent to (6) in $c - s$ space, the comparative static effects of n and b on c and s can be used to describe the comparative static effects of α on y and x. Thus, the sign of $\partial y^*/\partial\alpha$ is ambiguous if the intersection of the xx and yy loci is to right of the maximum point of the xx locus. If the intersection is to the left, the sign is negative if the equilibrium is saddle-point stable, and positive if the equilibrium is an unstable node. In contrast, the sign of $\partial x^*/\partial\alpha$ is always negative. These results imply that after a rise in α, s initially decreases, while c initially decreases or increases. Therefore, welfare may initially rise or fall. This transitional dynamic differs from the Ramsey model and its extensions to non-renewable resources (for example, Stiglitz, 1974). When α rises in the standard case, c and s increase smoothly from $t = 0$.

The effect of technological progress in the model is quite striking, suggesting that even technology changes that would seem presumptively propitious—technology increasing exogenously at an exponential rate—could exert negative short-run welfare effects. Since the 'short-run' is not clearly defined by theory, the possibility cannot be ruled out that negative technology effects could persist for a relatively 'long period' benchmarked against a typical planning horizon for humans. The discussion also raises the question: what is the potential for bias in the forecasts of models whose structures do not accurately reflect dynamic complexity? For example, what is the bias potential of Ramsey-based computable general equilibrium models currently used to forecast the economic effects of climate change or climate policies?

Finally, technological progress in the model may not close the welfare gap between two economies with the same parameters, implying that multiple balanced growth paths are possible in the system represented in equation (29). That is, economies with the same parameter configuration may not necessarily converge to the same balanced growth path in the long run in the model with nature capital in the utility function.[18] This result can be distinguished from the standard model with only one equilibrium in the $x - y$ plane and, consequently, only one balanced growth path in the $c - s$ plane

The possibility of multiple growth paths has implications for sustainable development. First, it adds another dimension to the claim that the criterion of non-declining welfare alone, which is sometimes cited as an indicator for sustainable development, is not a sufficient policy-making criterion (see, for example, Toman, Pezzey, and Krautkraemer, 1995). This is so since it is possible to have a number of sustainable paths with different welfare levels in the same economy, or to have different welfare paths among countries with the same configuration of economic parameters. Second, in contrast to the standard formulation, differences in initial conditions, reflecting historical accident or institutional factors, could persist in the form of long-run disparities in welfare levels, regardless of global technical advance. This in turn suggests a greater role for policy making to achieve desired welfare outcomes. Freely traded technology that causes the global rate of technology advance to converge across countries may not be sufficient to cause a convergence in welfare levels. Wealth transfers from rich to poor countries may be needed to encourage a higher welfare, sustained development path of poorer economies.

5 Conclusion

Multiple equilibria are common in ecological models (see, for example, May, 1977; Ludwig, Walker, and Holling, 1997). The associated dynamic complexity of such models and its economic implications have become the focus of an emerging literature in ecological economics (for example, Perrings and Walker, 1997; Arrow *et al.*, 2000; Mäler, 2000). Although the concept of multiple equilibria is sometimes used in economics (for example, in game theoretic models), the issue has not received much attention in the mainstream economic growth literature, nor in its endogenous technological progress and environmental extensions. Our results suggest the benefit of bringing a more 'systems ecological' perspective to the task of economic growth modeling, in the sense of making systems dynamics a primary modeling focus. Within this context, the 'ecological economics' terminology could be construed as the notion that the principles of systems modeling in ecology could be usefully extended to the study of economic growth. This task could prove to be fruitful in view of the possibility that the dynamic complexity in the economic component of the system would reinforce the dynamic complexity injected into the system by its ecological component.

Dynamic diversity arises in this paper through the channel of a mini-

[18] Comparing countries within the context of a single economy model is a common practice in the economic growth literature. However, a two-country framework, such as a North–South model, would offer the advantage of allowing the focus of the analysis to be extended to additional topics, such as trade or differential factor endowments.

mally defined quality-of-life measure, suggesting that dynamic diversity inherently may be a systemic property that captures the distinction between 'growth' and 'development', as these terms have been distinguished in the sustainable development literature. However, it seems likely that dynamic diversity could also arise through other channels. Endogenous population growth, for example, can add multiple equilibria in the economic growth context (for example, Nelson, 1956; Sato and Davis, 1971; Krutilla and Reuveny, 2000a).[19] Multiple equilibria might also arise from relaxing other assumptions in the paper, for example, modeling nature capital as a public good or disaggregating nature capital into components with different growth functions, including those with non-convexities (for example, Ludwig, Walker, and Holling, 1997; Mäler, 2000). Endogenizing other feedback loops not included in this paper, for example, the pollution impact on production, could have similar effects. Extending the model to incorporate these factors qualifies as worthy research extensions.

A second research extension could investigate the sensitivity of the model to different normative criteria for welfare specification. The non-declining welfare specification implicit in our paper and the Ramsey literature is relatively stringent. It would be interesting, for example, to see how results would differ if the definition of sustainability would require a positive time derivative of the entire value function (Dasgupta and Mäler, 2000). It also would be interesting to see how the dynamics would change when welfare is specified according to Chichilnisky's Criterion (a weighted measure of standard utility plus the value of utility as time goes to infinity) or when hyperbolic or other types of discount function are used (Heal, 1998).

A third set of research extensions would be to include physical capital in the model's production process, or to extend the model to include endogenous generation of technological progress. Finally, it would be interesting to simply link the current model with an existing purely ecological model in the literature that itself featured multiple steady states.

Although one would expect a number of specific conclusions to change with modeling extensions of the type suggested here, it does not seem likely that further research will turn around the central conclusion of the paper, that is, that complex dynamic behavior and possibly perverse, and policy-relevant, technology effects are likely to arise in economic growth models featuring only minor departures from the standard neoclassical assumptions. This possibility suggests the desirability of an expanded research agenda focusing on the manifestations and consequences of non-linear dynamic processes within the economic growth modeling context.

[19] Krutilla and Reuveny (2000b) show that resource extraction costs in a Ramsey framework can also generate multiple equilibria

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Appendix: Propositions and proofs

A1 Existence and stability of equilibria

The following notation is used: $c(s)_{cc}$ is the equation defining the *cc* locus; $c(s)_{ss}$ is the equation defining the *ss* locus; dc/ds (*cc*) is the slope of the *cc* locus; and dc/ds (*ss*) is the slope of the *ss* locus. The next two propositions establish the existence of simple or multiple equilibria.

Proposition 1

For $\rho > 0$, *there is at least one stationary state in the dynamic system.*

Proof

Equation (14) implies $s' < s_{msy}$ for $\rho > 0$ since the expression $g(s') - (b + n)$ is the slope of the *ss* locus at s'. With $c(s')_{cc} \equiv 0$ and $c(s')_{ss} > 0$, it follows that $c(s')_{cc} < c(s')_{ss}$. Given this interior origination point of the *cc* locus, the concavity of the *ss* locus, and the fact (see (15)) that dc/ds $(cc) > 0$, it follows that the *cc* locus must intersect the *ss* locus at least once from below. Recall also that the assumed transversality condition and assumption $U_c(0) = \infty$ imply that extinction is not optimal.

Proposition 2

There can only be one equilibrium on the domain $[s_{msy}, s^+)$. *At this equilibrium,* $dc/ds(cc) > dc/ds(ss)$. *Multiple equilibria require at least one equilibrium on* $(0, s_{msy})$.

Proof

The *cc* locus must intersect the ss locus from below on (s_{msy}, s^+) since $c(s')_{cc} < c(s')_{ss}$, dc/ds $(cc) > 0$, and dc/ds $(ss) \leqq 0$ for (s_{msy}, s^+). Let s_1 denote such an intersection point. With dc/ds $(cc) > dc/ds$ (ss), $c(s)_{cc} > c(s)_{ss}$ for $s > s_1$. Hence, there cannot be another point of intersection or tangency defining an equilibrium on (s_{msy}, s^+), implying that multiple equilibria can only arise with at least one intersection or tangency on $(0, s_{msy})$.

Note that multiple equilibria imply a 'sufficiently high' ρ. With ρ close to 0, equation (14) implies s' is close to s_{msy} and with dc/ds $(cc) > 0$, it is more likely the *cc* locus will uniquely intersect the *ss* locus on (s_{msy}, s^+) from this starting point.

A2 Comparative statics

Around saddle-point stationary states

We first state and prove propositions on the signs of the comparative statics around the saddle-point stable steady states.

Proposition 3

Assuming a saddle-point stable stationary state, $\partial c/\partial n < 0$ and $\partial c/\partial b < 0$ *on the domain* $(0, s_{msy})$, *while the signs of* $\partial c/\partial n$ and $\partial c/\partial b$ *are ambiguous on* (s_{msy}, s^+).

Proof

For a saddle-point stable steady state, $\det(J) < 0$; hence, the denominator of (18) is negative. The sign of the numerator is positive when $g_s - (b + n) \geqslant 0$, which is the *ss* slope which obtains when $s \in (0, s_{msy})$. This sign is ambiguous when $g_s - (b + n) < 0$, which is the *ss* slope condition which obtains when $s \in (s_{msy}, s^+)$. Hence, $\partial c/\partial n < 0$ and $\partial c/\partial b < 0$ for a steady state on $(0, s_{msy})$. The signs of these partial derivatives are ambiguous for a steady state on (s_{msy}, s^+).

Proposition 4

Assuming a saddle-point stable equilibrium, $\partial c/\partial \rho < 0$ *on the domain* $(0, s_{msy})$, $\partial c/\partial \rho = 0$ at $s_{msy,}$ and $\partial c/\partial \rho > 0$ *on the domain* $(s_{msy,}\ s^+)$.

Proof

For a saddle-point stable equilibrium, det $(J) < 0$; hence, the denominator of (19) is negative. The sign of the numerator is positive when $g_s - (b + n) > 0$, which is the *ss* slope which obtains when $s \in (0, s_{msy})$. The numerator is negative when $g_s - (b + n) < 0$, which is the *ss* slope condition which obtains when $s \in (s_{msy}, s^+)$. The numerator is 0 at s_{msy} where $g_s - (b + n) = 0$.

Note that proposition 4 differs from proposition 3 because the *ss* locus is unaffected by parametric changes in the discount rate.

Proposition 5

Assuming a saddle-point stable stationary state, $\partial s/\partial n < 0$ *and* $\partial s/\partial b < 0$.

Proof

The numerator in (20) is unambiguously positive since the first term, $s(\rho + b + n - g_s)$ is positive by (12), and the second term is also positive. The denominator is negative at a saddle-point stable stationary state. Hence, $\partial s/\partial n < 0$ and $\partial s/\partial b < 0$.

Proposition 6

Assuming a saddle-point stable stationary state, $\partial s/\partial \rho < 0$.

Proof

The numerator in (21) is positive. The denominator is negative at a saddle point. Hence, $\partial s/\partial \rho < 0$.

Around unstable nodes

Turning to the signs of the comparative statics around the unstable steady states, note that unstable stationary states can only occur on the domain $(0, s_{msy})$, since the condition for unstable nodes, dc/ds (*cc*) $< dc/ds$ (*ss*) cannot hold on $(s_{msy}\ ,s^+)$ with dc/ds (*cc*) > 0 and dc/ds (*ss*) less than 0 in that range. Second, the signs of the comparative statics referred to in propositions 3–6 are all unambiguous on $(0, s_{msy})$. Since the sign of det(J) in the denominators of (18)–(21) are reversed for the unstable stationary states, the signs for the unstable stationary states will be the opposite of the signs for the stable stationary states. Thus, the following can be asserted.

Proposition 7

At an unstable node, $\partial c/\partial \rho > 0$, $\partial c/\partial n > 0$ *and* $\partial c/\partial b > 0$.

Proposition 8
At an unstable node, $\partial s/\partial \rho > 0$, $\partial s/\partial n > 0$ *and* $\partial s/\partial b > 0$.

A3 Comparative dynamics and growth paths
This part of the appendix states and proves propositions about the model with labor-augmenting exogenous technological progress.

Proposition 9
The system represented in (29) exhibits a steady growth path in the $c - s$ *plane in which s and c constantly grow over time at the rate of technological progress*

Proof
Denote the x,y coordinates of the stationary equilibrium in the $x - y$ plane as x^* and y^*, respectively, and note that $x^* = s/A$ and $y^* = c/A$ implying that $s = Ax^*$ and $c = Ay^*$. Hence, c and s evolve over time according to $A = e^{\alpha t}$.

Proposition 10
After a rise in α, *s initially decreases and c initially decreases or increases.*

Proof
Along a balanced growth path, $s(t) = x^* e^{\alpha t}$ and $c(t) = y^* e^{\alpha t}$; thus, the magnitude of c and s at each point in time is determined both by the magnitudes of x^* or y^*—and the magnitude of $e^{\alpha t}$. Since the sign of $\partial x^*/\partial \alpha$ is negative and the sign of $\partial y^*/\partial \alpha$ is ambiguous, the negative effects of x^* on s and the possibly negative effect of y^* on c may dominate the positive effect of $e^{\alpha t}$ for small t. Over the long term, the exponential term will dominate and c and s will unambiguously rise at the rate of technological progress.

Proposition 11
Technological progress may not close the welfare gap between two economies with the same parameters, implying that multiple balanced growth paths are possible in the system represented in equation (29).

Proof
The possibility of multiple equilibria in the $x - y$ plane implies that the time paths of the economies in the $c - s$ plane cannot intersect if they start from different steady states in the $x - y$ plane and face the same rate of technological progress. Hence, multiple balanced growth paths are possible with the same parameter configuration.